

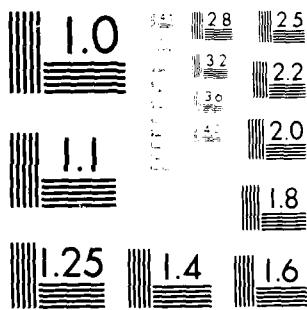
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ESTIMATION OF THE THRESHOLD IN FATIGUE CRACK

INITIATION MODELS: A BAYESIAN APPROACH

by

Nozer D. Singpurwalla  
Refik Soyer

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25 October 1983

The George Washington University  
School of Engineering and Applied Science  
Institute for Reliability and Risk Analysis

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THE GEORGE WASHINGTON UNIVERSITY  
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Abstract  
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25 October 1983

ESTIMATION OF THE THRESHOLD IN FATIGUE CRACK

INITIATION MODELS: A BAYESIAN APPROACH

by

Nozer D. Singpurwalla  
Refik Soyer

A Bayesian approach for the estimation of the threshold parameter in a probability model for fatigue crack failures is presented. The Weibull and the lognormal distributions are considered as suitable models for the number of cycles to crack initiation. The approach is discussed for the two models, and real life data from aircraft engine disks is analyzed.

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1. INTRODUCTION

The problem of selecting a suitable probability model for the number of cycles to crack initiation was discussed by Soyer (1982), (1983). There it was reported that the Weibull and the lognormal distributions were appropriate models for describing the distribution of the number of cycles to crack initiation.

In this report, a related problem, namely, the estimation of the threshold parameter in these life models, is considered. The *threshold parameter* is the minimum value that a random variable can take. For fatigue crack problems it is the largest number of cycles before which crack initiation is impossible.

In this paper, we discuss a Bayesian approach for estimating the threshold parameter of the Weibull and the lognormal models.

2. ESTIMATION OF THE THRESHOLD PARAMETER  
OF A WEIBULL DISTRIBUTION

The three-parameter Weibull distribution has the density function

$$f(x) = \frac{\beta}{\theta} \left( \frac{x-\phi}{\theta} \right)^{\beta-1} e^{-\left( \frac{x-\phi}{\theta} \right)^\beta}, \quad \phi, \beta, \theta > 0, x \geq \phi, \quad (2.1)$$

where  $\beta$  and  $\theta$  are the *shape* and the *scale* parameters, respectively, and  $\phi$  is the *threshold* parameter.

In this report, we shall, following the Bayesian paradigm, assume that prior information concerning the number of cycles to crack initiation exists, and that such information can be translated into prior distributions for the parameters of (2.1).

Let  $X$  be a random variable having the density (2.1). The distribution of  $X$  is dependent on the unknown parameters  $\phi$ ,  $\beta$ , and  $\theta$ . If  $h(\phi, \beta, \theta)$  denotes the joint prior distribution of the parameters, then given  $n$  lifetimes  $\underline{x} = (x_1, \dots, x_n)$  and  $\ell(\phi, \beta, \theta; \underline{x})$  the likelihood function, by Bayes' theorem the posterior distribution  $g(\phi, \beta, \theta | \underline{x})$  is proportional to  $h(\phi, \beta, \theta) \cdot \ell(\phi, \beta, \theta; \underline{x})$ , the constant of proportionality being independent of the parameters. If  $\beta$  is an element of the parameter space  $B$  and  $\theta$  an element of the space  $(H)$ , then the posterior distribution of  $\phi$  can be obtained as

$$g_1(\phi | \underline{x}) \propto \int_B \int_{(H)} g(\phi, \beta, \theta | \underline{x}) d\theta d\beta. \quad (2.2)$$

The *modal* value of the above posterior distribution can be used as a Bayes estimate of the threshold parameter  $\phi$ . This value is also known as the *generalized maximum likelihood estimator*, and is viewed as a Bayes estimator when a particular loss function is not specified.

The aforesited analysis poses several difficulties, because there exists no closed form solution for the integration in (2.2); thus numerical integration techniques will have to be used.

We will consider two different cases. In the first case, independent prior distributions will be assumed for all three parameters. In the second case, we will consider a joint prior distribution for  $\phi$  and  $\theta$ , and a prior distribution for  $\beta$ , which is independent of the joint distribution of  $\phi$  and  $\theta$ .

### 2.1 Independent Priors on All Parameters

Assume that a sample of  $n$  lifetimes is observed from a Weibull distribution having the density (2.1). The likelihood function is

$$l(\phi, \beta, \theta; \underline{x}) = \left[ \frac{\beta}{\theta} \right]^n \left\{ \prod_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^{\beta-1} \right\} e^{-\sum_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^\beta}. \quad (2.3)$$

We assume a uniform prior over  $(0, M)$  for the threshold parameter  $\phi$ ; thus the probability density function for  $\phi$  is:

$$h_1(\phi) = \frac{1}{M}, \quad 0 \leq \phi \leq M.$$

Similarly, the prior on  $\beta$  is assumed to be uniform over  $(M_1, M_2)$ , giving

$$h_2(\beta) = \frac{1}{M_2 - M_1}, \quad M_1 \leq \beta \leq M_2,$$

as its probability density function.

For the scale parameter  $\theta$ , we consider a four parameter beta density, so that

$$h_3(\theta) = \frac{\Gamma(g_1+g_2)}{\Gamma(g_1)\Gamma(g_2)} \frac{(g_1-1)^{g_1-1} (g_2-1)^{g_2-1}}{(b-a)^{g_1+g_2-1}}, \quad (2.3.1)$$

where  $a \leq \theta \leq b$ ,  $g_1, g_2 > 0$ .

Because of independence, the joint prior distribution of  $\phi$ ,  $\beta$ , and  $\theta$  can be written as the product of the marginal densities,

$$h(\phi, \beta, \theta) = h_1(\phi)h_2(\beta)h_3(\theta). \quad (2.4)$$

By using Bayes' theorem the joint posterior density of  $\phi$ ,  $\beta$ , and  $\theta$  is:

$$g(\phi, \beta, \theta | \underline{x})$$

$$\propto \left( \frac{\beta}{\theta} \right)^n \left\{ \prod_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^{\beta-1} \right\} e^{-\sum_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^\beta} (\theta-a)^{g_1-1} (b-\theta)^{g_2-1}. \quad (2.5)$$

In order to obtain the posterior distribution of  $\phi$  the double integral with respect to  $\beta$  and  $\theta$  given below should be evaluated:

$$g_1(\phi | \underline{x}) \propto \int_{M_1}^{M_2} \int_a^b \left( \frac{\beta}{\theta} \right)^n \left\{ \prod_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^{\beta-1} \right\} e^{-\sum_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^\beta} (\theta-a)^{g_1-1} (b-\theta)^{g_2-1} d\theta d\beta. \quad (2.6)$$

There exists no closed form solution for the above integral; thus a numerical technique is used to evaluate it.

This approach is applied to some engine disk data presented in the appendix. A uniform prior density is assumed for the threshold parameter  $\phi$ , over  $(0, 700)$ ; this density is indicated in Figure 1. The

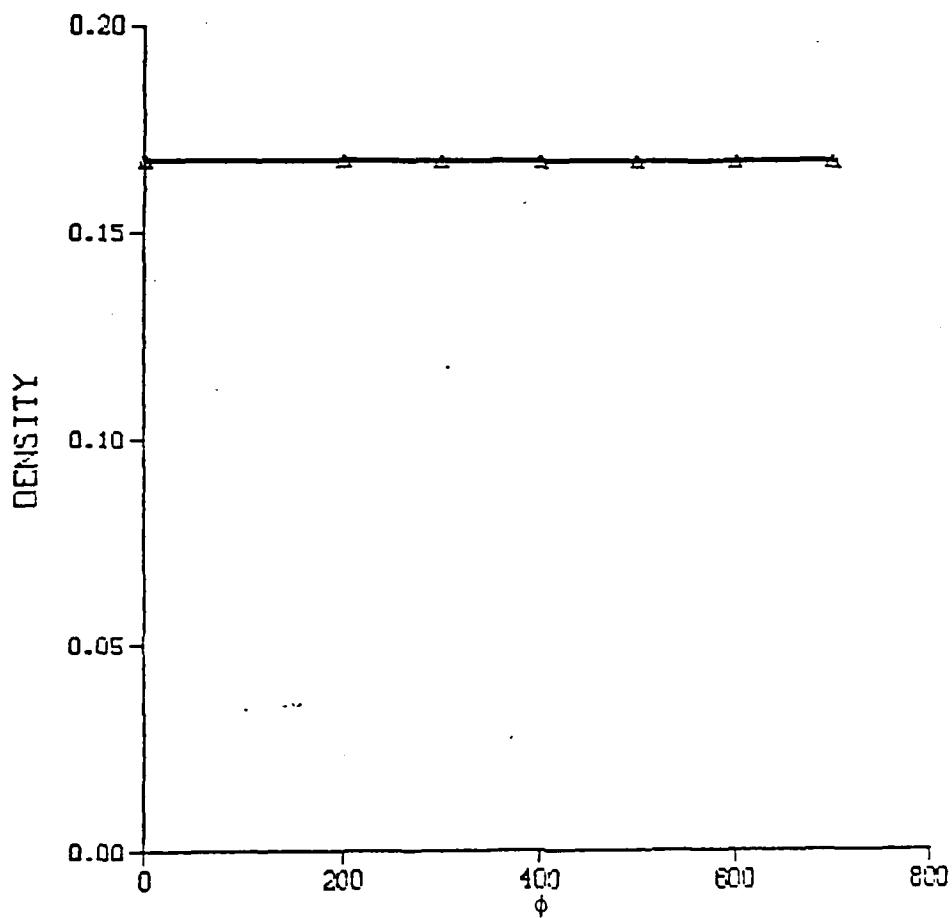


Figure 1. Uniform prior density for threshold parameter  $\phi$ .

prior for  $\beta$  is assumed to be a uniform over the interval (1,6), and a beta prior density over the interval (200,900) is considered for  $\theta$ , with  $g_1 = 3$  and  $g_2 = 2$ ; see Equation (2.3.1). The integral in (2.6) is solved by using Simpson's rule and the posterior distribution of  $\phi$  obtained from (2.6) is shown in Figure 2. Note that although we started with a uniform prior on  $\phi$ , the data have enabled us to revise our prior evaluation, to yield a posterior distribution which has a mode at 375 equivalent flight hours (EFH). This value may be used as the Bayes estimate of the threshold parameter.

The assumption of independent priors on all the parameters may not be realistic; we drop this assumption in the next section.

## 2.2 A Joint Prior on the Threshold and the Scale Parameters

In this section a joint prior distribution is considered for the parameters  $\phi$  and  $\theta$ , and an independent prior density is considered for  $\beta$ .

The joint prior distribution considered here was first suggested by Varde (1969) for the two parameter exponential distribution. This joint density has the form:

$$h_2(\phi, \theta) \propto \frac{1}{\theta^{v+1}} e^{-(\mu-\lambda\phi)/\theta} \cdot \delta(\eta-\phi) , \quad (2.7)$$

where

$$\delta(u) = \begin{cases} 1, & u \geq 0 \\ 0, & u < 0 ; \end{cases}$$

$$\phi \geq 0 , \text{ and } \theta > 0 .$$

The parameters  $v$  and  $\lambda$  are positive integers such that  $v \leq \lambda < \mu/\eta$ ;  $\mu > 0$ ,  $\mu$  and  $\eta$  are such that  $\eta > 0$ . Note that when

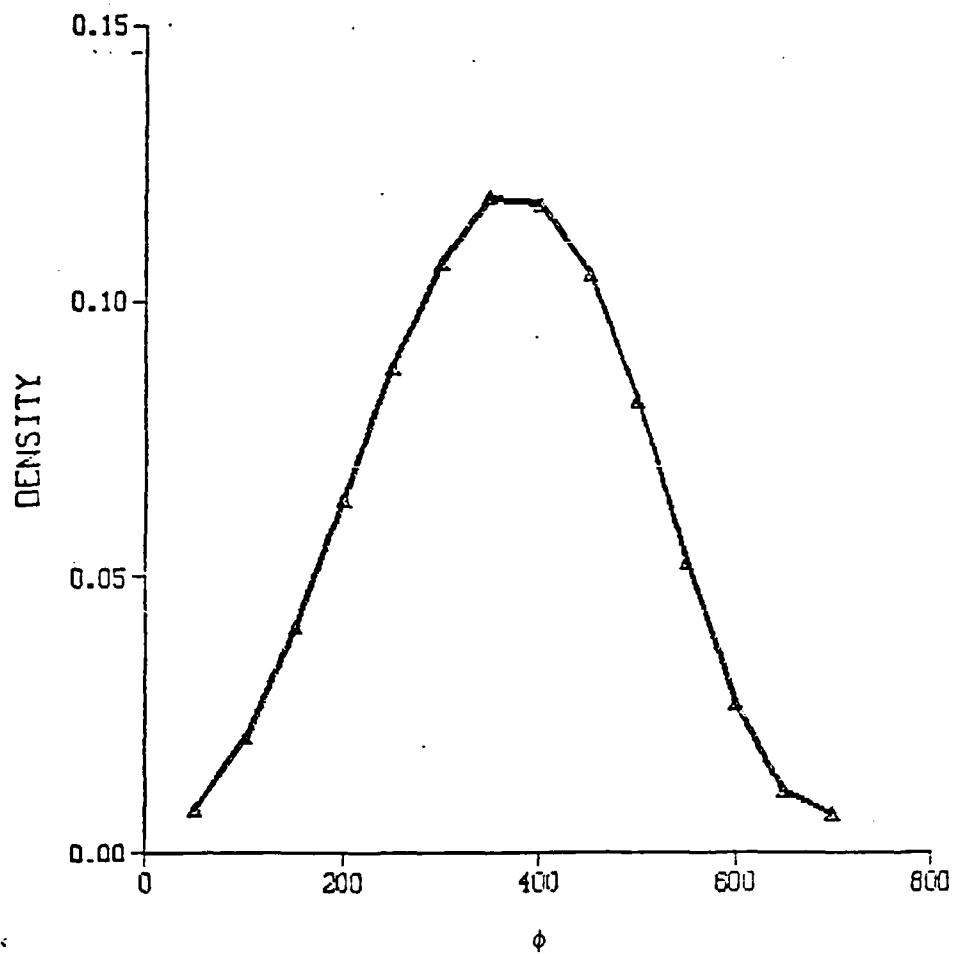


Figure 2. Posterior density of threshold parameter  $\phi$ .

$\lambda = \mu = 0$  and  $\eta \rightarrow \infty$ , (2.7) is equivalent to assuming independent priors on  $\phi$  and  $\theta$ , with the prior of  $\phi$  being *diffuse* and the prior of  $\theta$  being proportional to  $1/\theta^{v+1}$ .

A plot of this joint prior density for  $v = 1$ ,  $\eta = 700$ ,  $\mu = 6500$ , and  $\lambda = 3$  is shown in Figure 3.

If we integrate (2.7) with respect to  $\theta$ , the marginal prior for the threshold parameter is obtained as

$$h_1(\phi) \propto \frac{\Gamma(v)}{(\mu - \lambda\phi)^v} \delta(\eta - \phi) . \quad (2.8)$$

A plot of this marginal prior density is shown in Figure 4. As can be seen from this figure, the prior density increases with  $\phi$ . The parameter  $\lambda$  controls the slope of the density function. For small values of  $\lambda$  the prior becomes flatter.

The parameter  $\eta$  determines the range of values of  $\phi$  over which its density is nonzero; that is,  $\eta$  is the largest value that  $\phi$  can take.

The parameters  $\mu$  and  $v$  are harder to interpret, but once the parameters  $\eta$  and  $\lambda$  are determined, the inequality  $v \leq \lambda < \mu/\eta$  enables us to specify the values of these parameters.

An independent uniform prior density on  $(M_1, M_2)$  is assumed for the shape parameter  $\beta$ , and thus the joint prior on  $\phi$ ,  $\beta$ , and  $\theta$  can be written as:

$$h(\phi, \beta, \theta) \propto \frac{1}{\theta^{v+1}} e^{-(\mu - \lambda\phi)/\theta} \delta(\eta - \phi) \frac{1}{M_2 - M_1} . \quad (2.9)$$

By Bayes' theorem the joint posterior density of the three parameters is obtained as:

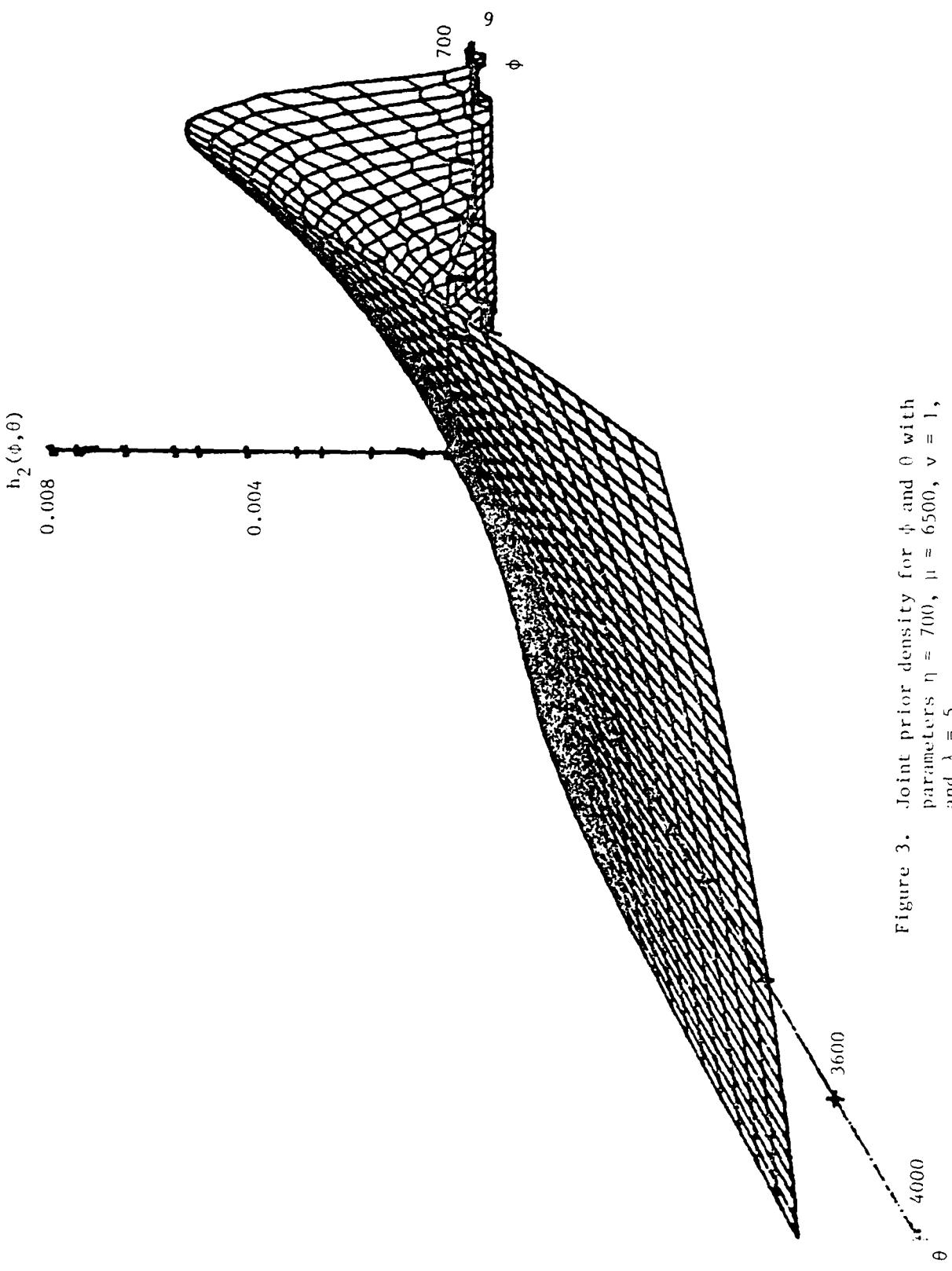


Figure 3. Joint prior density for  $\phi$  and  $\theta$  with parameters  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 1$ , and  $\lambda = 5$ .

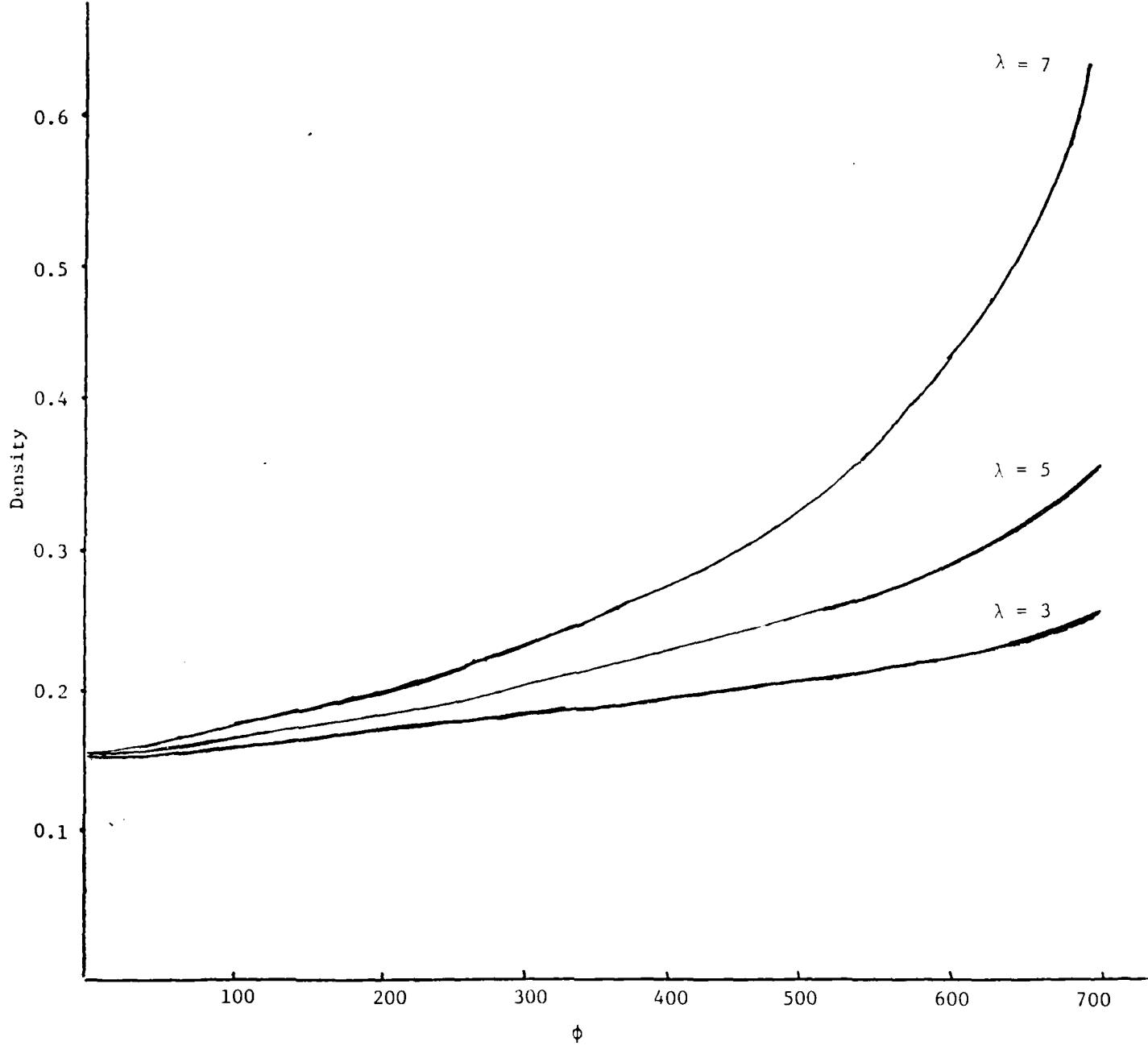


Figure 4. Marginal prior density for threshold parameter  $\phi$  with  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 1$ , and  $\lambda = 3, 5$ , and  $7$ .

$$g(\phi, \beta, \theta | \underline{x})$$

$$\propto \left( \frac{\beta}{\theta} \right)^n \left\{ \prod_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^{\beta-1} \right\} e^{-\sum_{i=1}^n \left( \frac{x_i - \phi}{\theta} \right)^\beta} \frac{1}{\theta^{v+1}} e^{-(\mu - \lambda \phi)/\theta}, \quad (2.10)$$

$$0 \leq \phi \leq \eta, \theta > 0, M_1 \leq \beta \leq M_2.$$

To obtain the marginal posterior distribution of  $\phi$ , we need to integrate (2.10) with respect to  $\beta$  and  $\theta$ . Here again, there is no closed form solution for the posterior distribution of  $\phi$ , and therefore the above integral is to be solved numerically.

This approach is applied to the engine disk data by using different forms of the joint prior (2.7).

The prior density for  $\beta$  is assumed to be uniform over the interval (1,6). The joint prior on  $\phi$  and  $\theta$  will have the parameters  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 1$ , and  $\lambda = 5$  (see Figure 3). The marginal prior density for  $\phi$  is shown in Figure 5. The posterior distribution of  $\phi$  is obtained by solving the double integral in (2.10) numerically with respect to  $\theta$  and  $\beta$ . The posterior distribution of  $\phi$  is presented in Figure 6. The modal value is at 500 EFH.

Another joint prior considered for  $\phi$  and  $\theta$  has parameters  $\eta \approx 700$ ,  $\mu = 6500$ ,  $v = 3$ , and  $\lambda = 5$ . The prior on  $\beta$  is again a uniform over (1,6). By using numerical integration in (2.10) the posterior distribution of  $\phi$  is obtained. The mode of the posterior distribution occurs at 600 EFH. This posterior distribution is shown in Figure 7. Another joint prior for  $\phi$  and  $\theta$  has parameters  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 2$ , and  $\lambda = 5$ . In this case the mode of the posterior occurs at 550 EFH (see Figure 8). Thus

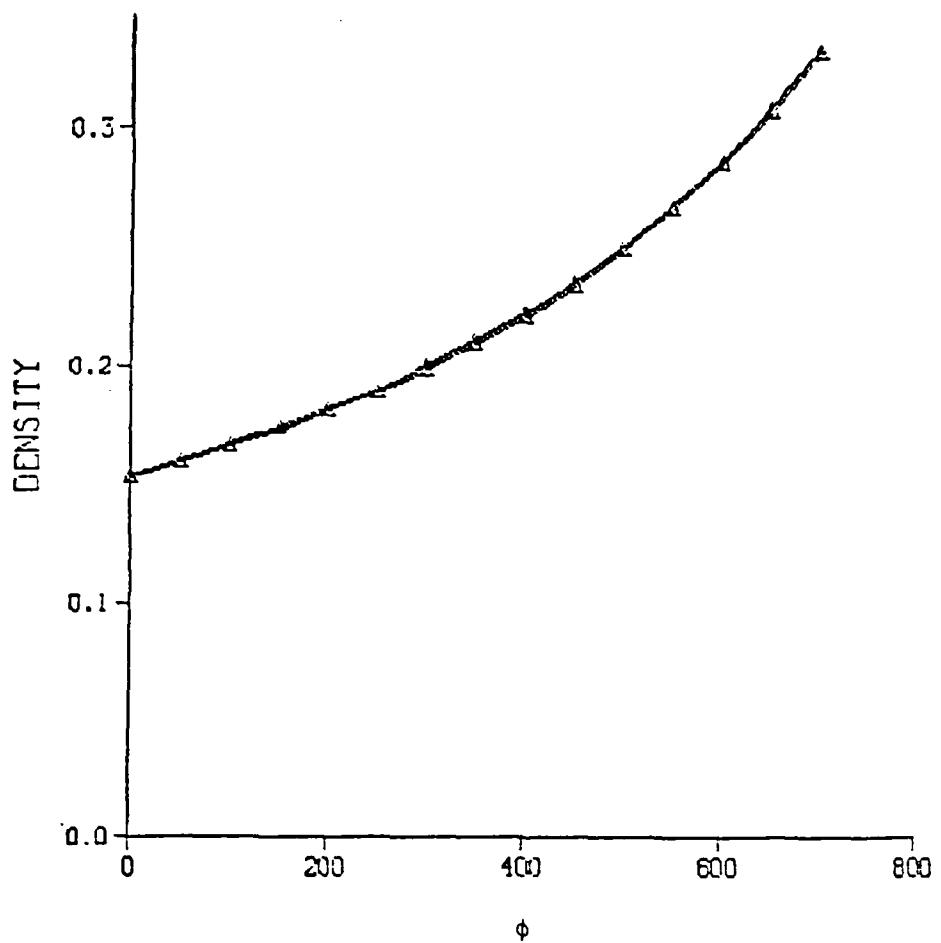


Figure 5. Marginal prior density for threshold parameter  $\phi$  with  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 1$ , and  $\lambda = 5$ .

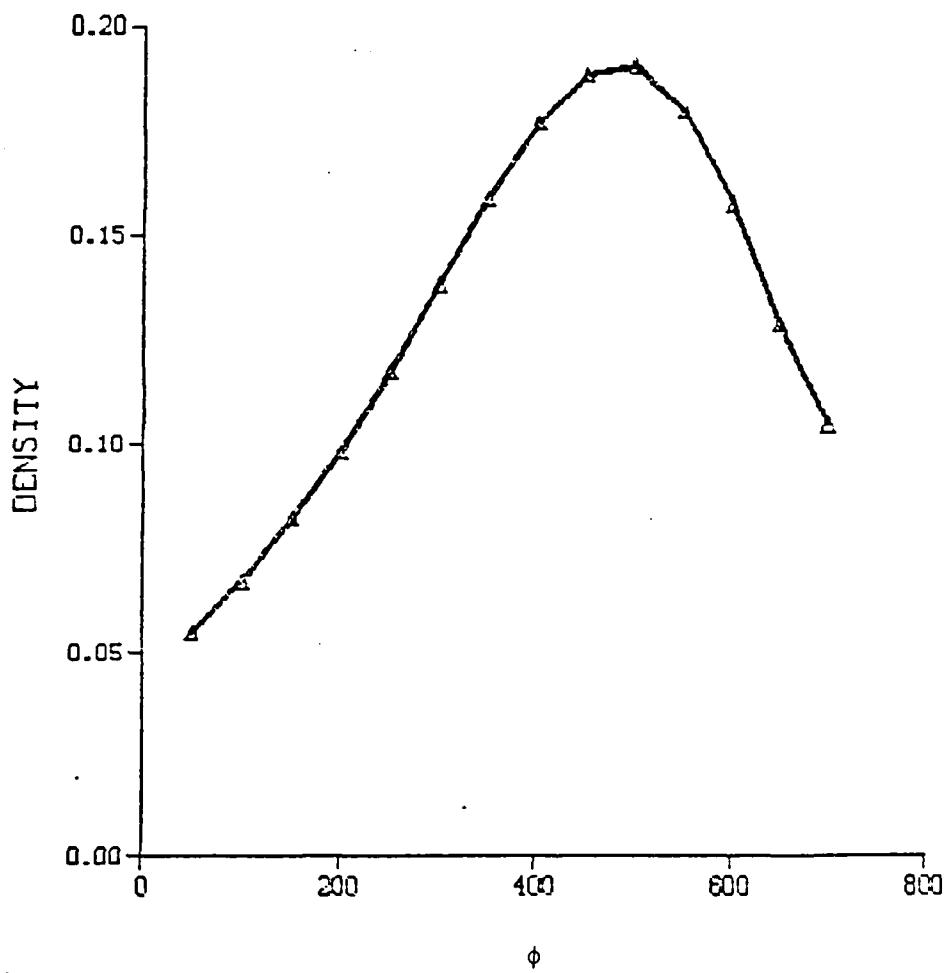


Figure 6. Marginal posterior density for threshold parameter  $\phi$  assuming the joint prior of Figure 3.

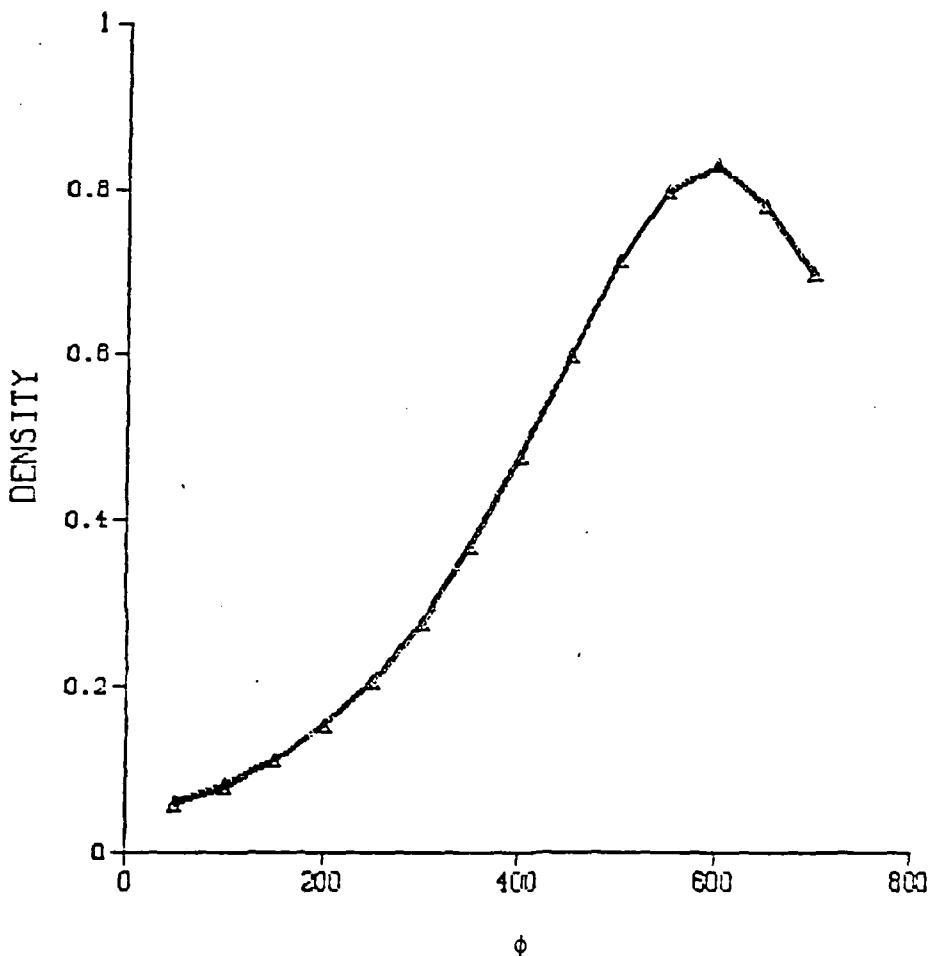


Figure 7. Marginal posterior density for threshold parameter  $\phi$  assuming a joint prior for  $\phi$  and  $\theta$  with  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 3$ , and  $\lambda = 5$ .

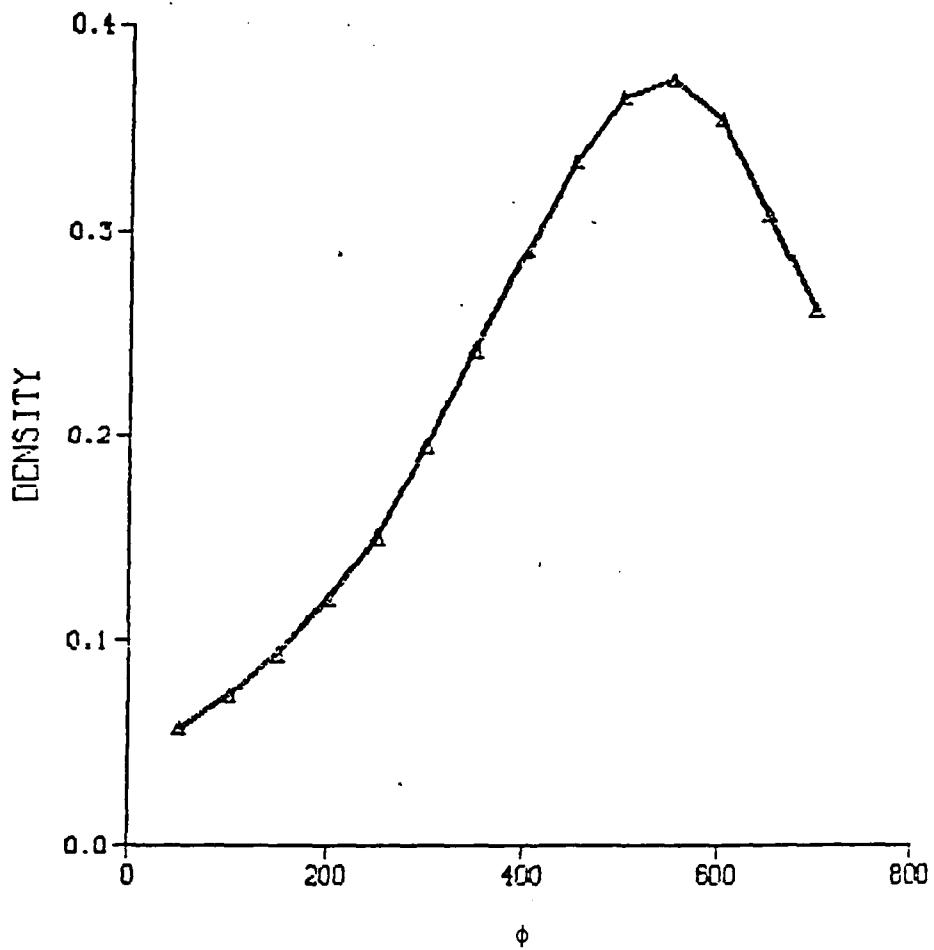


Figure 8. Marginal posterior density for threshold parameter  $\phi$  assuming a joint prior for  $\phi$  and  $\theta$  with  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 2$ , and  $\lambda = 5$ .

it appears that increasing the value of  $v$  shifts the mode of the posterior distribution closer to  $\eta$ .

In order to see the effect of a decrease in the value of the parameter  $\lambda$ , we consider the joint prior on  $\phi$  and  $\theta$  with parameters  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 2$ , and  $\lambda = 3$ . The posterior distribution of  $\phi$  is shown in Figure 9. The posterior mode occurs at 300 EFH. That is, a decrease in  $\lambda$ , with the other parameters held fixed, shifts the posterior mode closer to zero. This can be observed by comparing the modes in Figures 8 and 9.

Thus it appears that the posterior mode is sensitive to the values of the prior parameters. As we have observed, the slope of the marginal prior density of  $\phi$  can be controlled by parameter  $\lambda$ . Small values of  $\lambda$  give a flat prior density for  $\phi$ , whereas large values of  $\lambda$  give more weight to the higher threshold values.

### 3. ESTIMATION OF THE THRESHOLD PARAMETER OF A LOGNORMAL DISTRIBUTION

The three-parameter lognormal distribution has the density function:

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma(x-\phi)} e^{-\frac{1}{2} \left[ \frac{\ln(x-\phi)-\mu}{\sigma} \right]^2}, \quad x \geq \phi, \quad (3.1)$$

where  $\phi$  is the threshold parameter,  $\mu$  is the mean of the random variable  $Y = \ln(x-\phi)$ , and  $\sigma^2$  is the variance of  $Y$ . Given  $n$  lifetimes  $x = (x_1, \dots, x_n)$ , the likelihood function is:

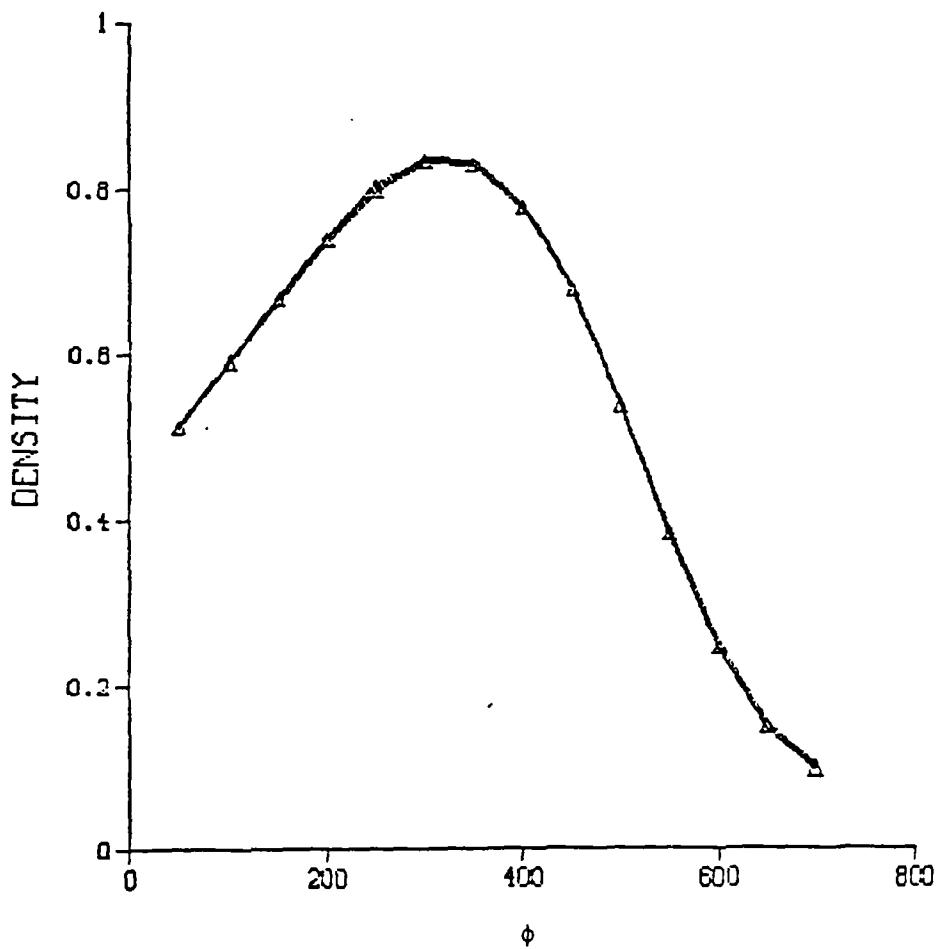


Figure 9. Marginal posterior density for threshold parameter  $\phi$  assuming a joint prior for  $\phi$  and  $\theta$  with  $\eta = 700$ ,  $\mu = 6500$ ,  $v = 2$ , and  $\lambda = 3$ .

$$\ell(\phi, \mu, \sigma^2; \underline{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \left\{ \prod_{i=1}^n \frac{1}{x_i - \phi} \right\}^{-\frac{1}{2}} \prod_{i=1}^n \left[ \frac{\ln(x_i - \phi) - \mu}{\sigma} \right]^2. \quad (3.2)$$

Some features of this likelihood function are discussed in Hill (1963). It is shown that there exist paths along which the likelihood function, for any sample, tends to  $+\infty$  as the triple  $(\phi, \mu, \sigma^2)$  approaches  $(x_{(1)}, -\infty, +\infty)$ , where  $x_{(1)}$  is the first-order statistic. Thus maximum likelihood estimation, if used here, would lead to difficulties. From a Bayesian point of view, there is no real problem, since the maximum of the likelihood function is not of interest.

The goal is to obtain the posterior distribution of the threshold parameter  $\phi$ , and use the mode of this distribution as the Bayes estimate of  $\phi$ .

Two different cases are considered in our analysis. In the first case, independent prior densities are assumed for all three parameters. In the second case, a joint prior density is assumed for  $\mu$  and  $\sigma^2$ , whereas a prior density which is independent of the joint distribution of  $\mu$  and  $\sigma^2$  is assumed for  $\phi$ .

### 3.1 Independent Priors on All Parameters

A uniform prior density over  $(0, M)$  is assumed for  $\sigma^2$ . For the time being, assume independent priors for  $\phi$  and  $\mu$ . Let  $h_1(\phi)$  and  $h_2(\mu)$  denote the prior densities for  $\phi$  and  $\mu$ , respectively. The likelihood function is given in (3.2). By using Bayes' theorem, the joint posterior density of  $\phi$ ,  $\mu$ , and  $\sigma^2$  is obtained as:

$$g(\phi, \mu, \sigma^2 | \underline{x})$$

$$\propto h_1(\phi) h_2(\mu) \left( \frac{1}{\sigma^2} \right)^{n/2} \left\{ \prod_{i=1}^n \frac{1}{(x_i - \phi)} \right\} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n [\ln(x_i - \phi) - \mu]^2}. \quad (3.3)$$

To obtain the posterior distribution of  $\phi$ , (3.3) is integrated with respect to  $\sigma^2$  and  $\mu$ . The integral of (3.3) with respect to  $\sigma^2$  can be obtained in closed form. The integral

$$\int_0^M \left( \frac{1}{\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n [\ln(x_i - \phi) - \mu]^2} d\sigma^2 \quad (3.4)$$

can be evaluated analytically for large  $M$ . For large values of  $M$  (3.4) becomes:

$$\left( \frac{1}{\frac{1}{2} \sum_{i=1}^n [\ln(x_i - \phi) - \mu]^2} \right)^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2} - 1\right). \quad (3.4.1)$$

To obtain the posterior distribution of  $\phi$ , the following integral needs to be evaluated:

$$g_1(\phi | \underline{x})$$

$$\propto \int h_1(\phi) h_2(\mu) \left\{ \prod_{i=1}^n \frac{1}{(x_i - \phi)} \right\} \left( \frac{1}{\sum_{i=1}^n [\ln(x_i - \phi) - \mu]^2} \right)^{\frac{n}{2}-1} d\mu. \quad (3.5)$$

There exists no closed form solution for (3.5), therefore it is evaluated by numerical integration.

This approach is applied to the engine disk data. A uniform prior density is assumed for  $\mu$  over  $(-100, 100)$ . The prior for  $\phi$  is a

beta density over (50,750) with parameters  $g_1 = 2$  and  $g_2 = 7$  [see Equation (2.3.1)]. The prior density of  $\phi$  is shown in Figure 10. The posterior distribution (3.5) is obtained by numerical integration. The mode of the posterior distribution is at 550 EFH. The posterior distribution of  $\phi$  is shown in Figure 11.

### 3.2 Joint Prior on $\mu$ and $\sigma^2$

The joint prior density which is considered for  $\mu$  and  $\sigma^2$  is suggested in Raiffa and Schlaifer (1968) and is called the normal-gamma density. This joint density has the following form:

$$h_{23}(\mu, \sigma^2) \propto \theta^{\frac{1}{2}(v'-1)} e^{-\frac{1}{2}\theta[n'(\mu-m')^2 + v'u']} , \quad (3.6)$$

where  $\theta = (1/\sigma^2)$ ,  $-\infty < m' < \infty$ ,  $u', n', v' > 0$ .

The joint prior density is the product of the densities:

$$h_2(\mu|\theta) \propto e^{-\frac{1}{2}\theta n'(\mu-m')^2}, \quad -\infty < \mu < \infty \quad (3.7)$$

and

$$h_3(\theta) \propto e^{-\frac{1}{2}\theta v'u'} \theta^{\frac{v'}{2}-1}, \quad \theta = \frac{1}{\sigma^2}, \quad (3.7.1)$$

where  $h_2(\mu|\theta)$  is a normal probability density and  $h_3(\theta)$  is a gamma density. If  $n' = 0$  in the joint density in (3.6), then  $\mu$  and  $\theta$  have independent priors. In this case,  $\mu$  will have a diffuse prior and  $\theta$  has a gamma prior. The marginal prior density for  $\mu$  is a student-t of the form:

$$h_2(\mu) \propto [v' + (\mu-m')^2 n'/u']^{-\frac{1}{2}(v'+1)}. \quad (3.7.2)$$

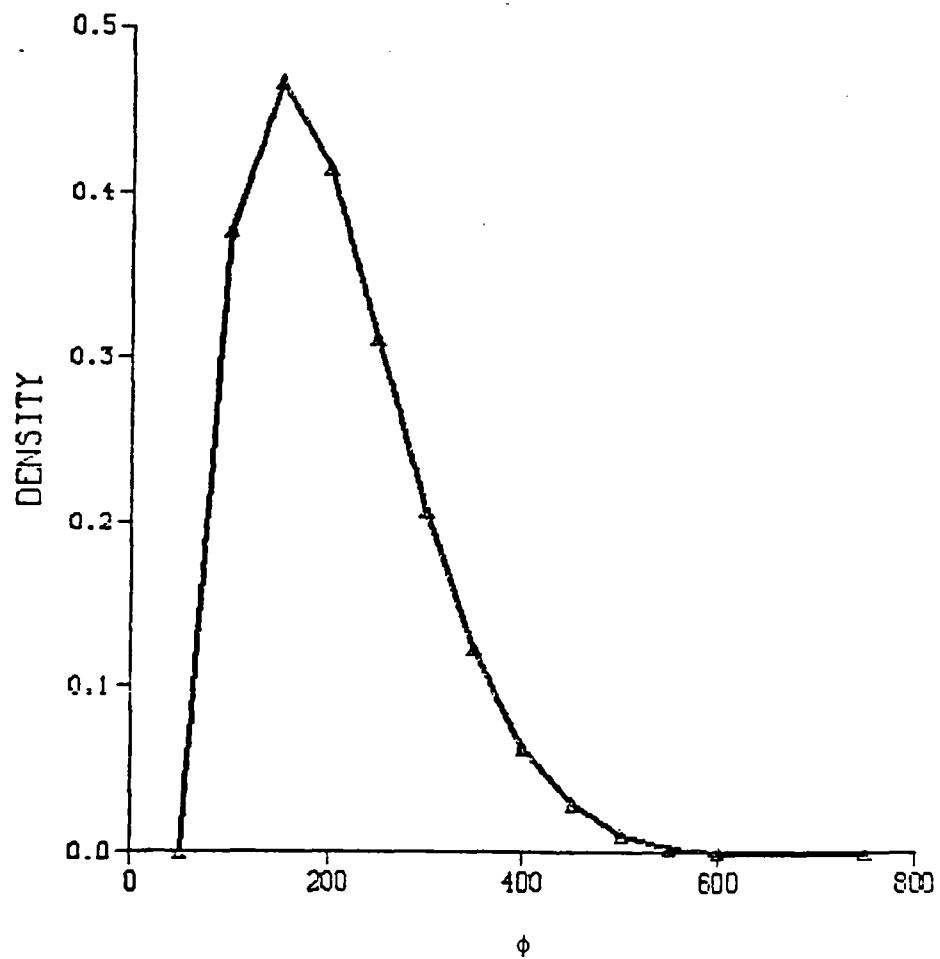


Figure 10. Beta prior density for threshold parameter  
 $\phi$  over  $(50, 750)$  with  $g_1 = 2$ ,  $g_2 = 7$ .

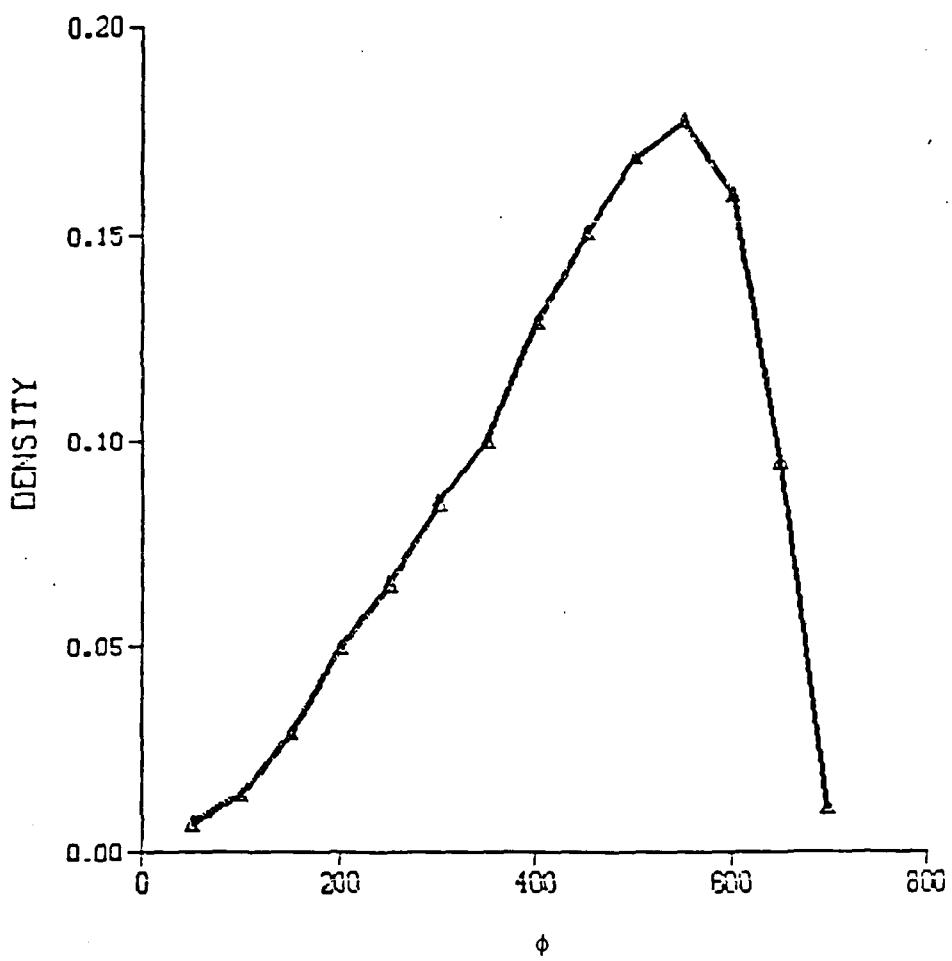


Figure 11. Marginal posterior density for threshold parameter assuming beta prior density for  $\phi$  over  $(50, 750)$  with  $g_1 = 2$ ,  $g_2 = 7$ .

The advantage of using joint prior density (3.6) is that it is mathematically tractable. It enables us to solve one of the integrals analytically in the evaluation of the posterior distribution of  $\phi$ .

Let us assume a prior density for  $\phi$  which is independent of the joint prior (3.6) and denote it by  $h_1(\phi)$ . By using Bayes' theorem, the joint posterior distribution of  $\phi$ ,  $\mu$ , and  $\theta$  is obtained as:

$$g(\phi, \mu, \theta | x) \quad (3.8)$$

$$\propto h_1(\phi) \left\{ \prod_{i=1}^n \frac{1}{(x_i - \phi)} \right\} \theta^{\frac{1}{2}(n+v'-1)} e^{-\frac{\theta}{2} \left[ n'(\mu-m')^2 + v'u' + \sum_{i=1}^n [\ln(x_i - \phi) - \mu]^2 \right]}.$$

To obtain the posterior distribution of  $\phi$ , (3.8) is integrated with respect to  $\theta$  and  $\mu$ . The integral with respect to  $\theta$  can be evaluated analytically and the joint posterior density of  $\phi$  and  $\mu$  is obtained as:

$$g_{12}(\phi, \mu | x) \quad (3.9)$$

$$\propto h_1(\phi) \left\{ \prod_{i=1}^n \frac{1}{(x_i - \phi)} \right\} \left\{ \frac{1}{\frac{1}{2} n'(\mu-m')^2 + v'u' + \sum_{i=1}^n [\ln(x_i - \phi) - \mu]^2} \right\}^{\frac{1}{2}(n'+v'+1)}.$$

The posterior distribution of  $\phi$  is obtained by integrating (3.9) numerically with respect to  $\mu$ .

This approach is applied to the engine disk data by assuming a beta prior density for  $\phi$ .

A joint prior of the form (3.6) with parameters  $v' = 9$ ,  $n' = 10$ ,  $m' = 4$ ,  $u' = 5$  is assumed for  $\sigma^2$  and  $\mu$ . A beta prior density over  $(50, 750)$  with parameters  $g_1 = 2$  and  $g_2 = 7$  is assumed for  $\phi$ . The posterior distribution of  $\phi$  is shown in Figure 12. The mode of the

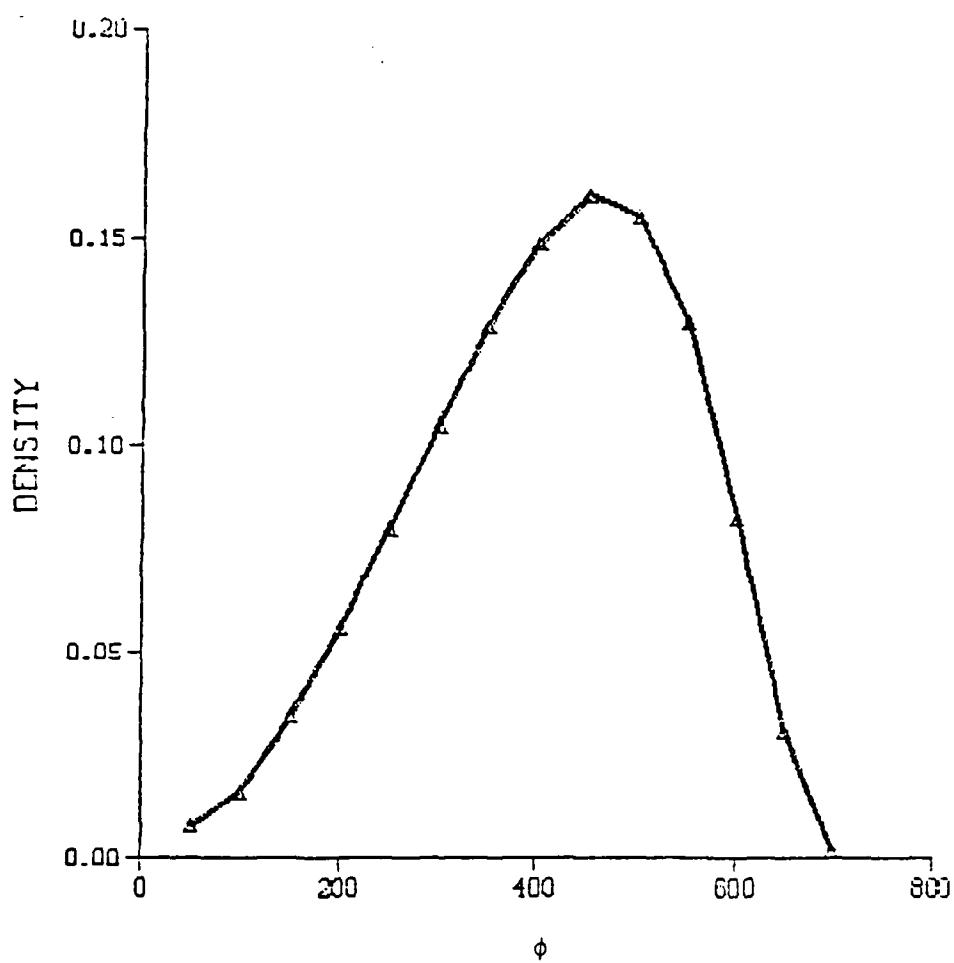


Figure 12. Marginal posterior density for threshold parameter  $\phi$  assuming a beta prior for  $\phi$  over  $(50, 750)$  with  $g_1 = 2$ ,  $g_2 = 7$ , and assuming a joint prior for  $\mu$  and  $\sigma^2$ .

posterior distribution is at 475 EFH. Note that for the independent priors in Section 3.1, the mode was 550 EFH.

Another beta prior over (50,750) with parameters  $g_1 = 2$  and  $g_2 = 5$  is assumed for  $\phi$ . The prior density is shown in Figure 13. The same joint prior is assumed for  $\mu$  and  $\sigma^2$ . By using numerical integration, the posterior distribution of  $\phi$  is obtained and is presented in Figure 14. The mode of the posterior distribution is at 600 EFH. Note that by assuming independent priors in Section 3.1 and assuming a joint prior for  $\mu$  and  $\sigma^2$  in this section, the Bayes estimates obtained for  $\phi$  are similar.

#### 4. CONCLUDING REMARKS

In this report we have attempted to present a Bayesian approach for the estimation of the threshold parameter in fatigue crack models. The nontrivial forms of the density functions have required use of numerical techniques to evaluate the posterior distributions. These numerical techniques appear to be efficient in evaluating the integrals. We have attempted to consider different forms of priors in our analysis. Sometimes computational difficulties have prevented us from considering other forms of prior densities.

On the basis of the analysis made, the range (400,550) EFH appears to be a region of high probability for the threshold parameter in engine disks. Availability of more information and selection of the prior parameters appropriately may improve these results.

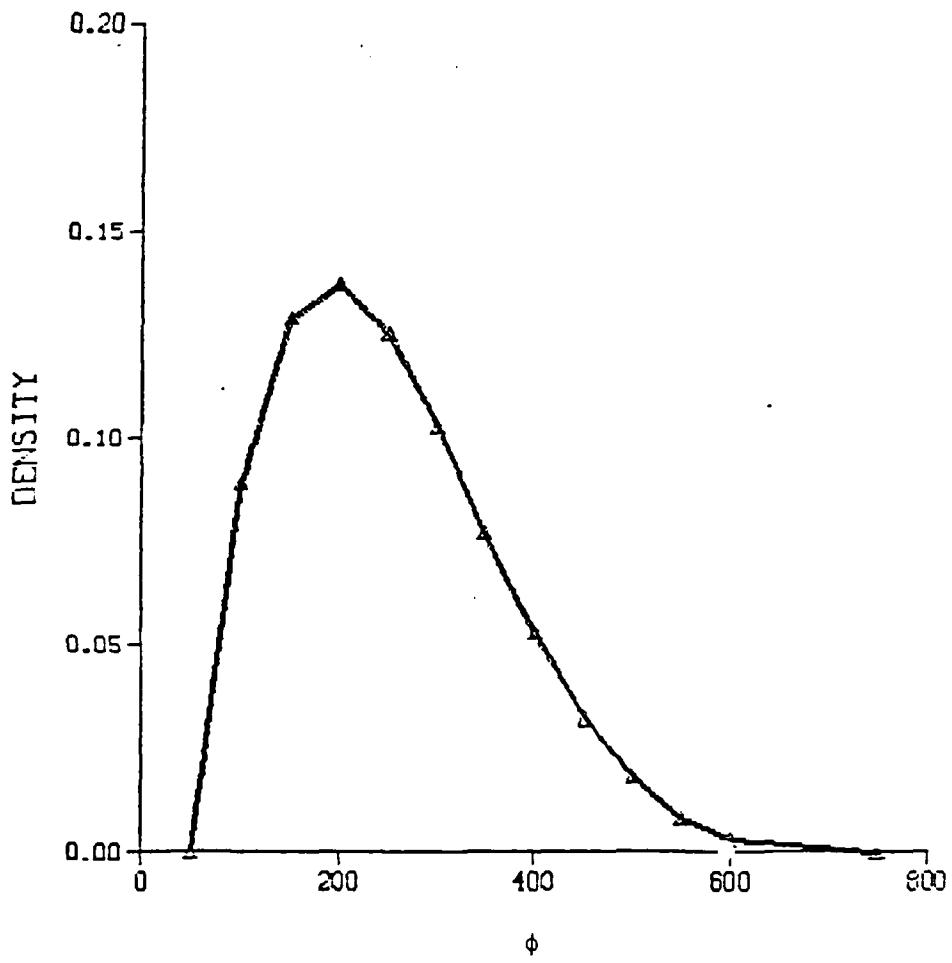


Figure 13. Beta prior density for threshold parameter  $\phi$  over (50, 750) with  $g_1 = 2$ ,  $g_2 = 5$ .

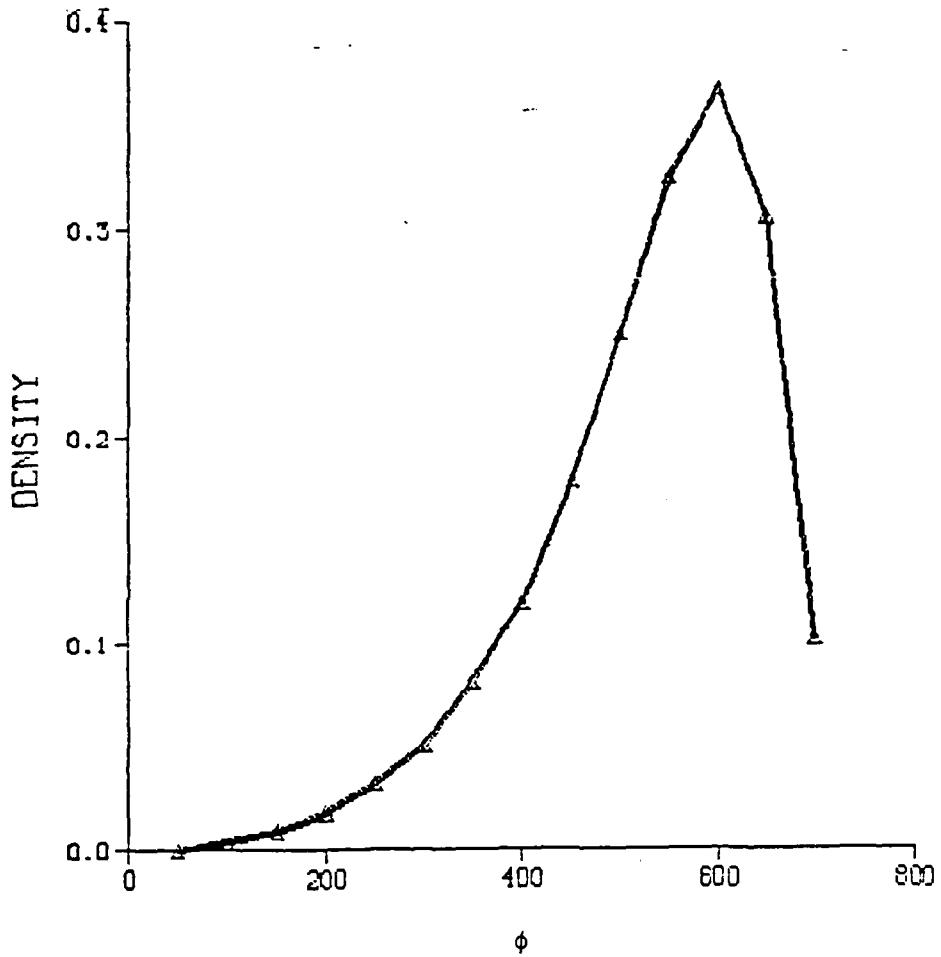


Figure 14. Marginal posterior density for threshold parameter  $\phi$  assuming beta prior for  $\phi$  over  $(50, 750)$  with  $g_1 = 2$ ,  $g_2 = 5$ , and assuming joint prior for  $\mu$  and  $\sigma^2$ .

## APPENDIX

Table A.1  
Crack Initiation in 10th Compressor Disk

Disk Number	Equivalent Flight Hours
1	1050
2	950
3	820
4	1050
5	780
6	910

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